THE FIRST HITTING DISTRIBUTION OF A SPHERE FOR SYMMETRIC STABLE PROCESSES

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1. **Introduction.** Let $X(t)$ be a symmetric stable process on N dimensional Euclidean space R^N, having exponent α and transition density

$$p(t, x) = (2\pi)^{-N} \int_{R^N} \exp [-t|\theta|^\alpha] e^{-i(\theta, x)} \, d\theta.$$

We will always work with the version of the process $X(t)$ which is a standard Markov process. (See Chapter 1 of [1] for a complete description of a standard process.) For any $r>0$ let $S_r=\{y \in R^N : |y|=r\}$ denote the sphere of center 0 and radius r. Set $T_r=\inf\{t>0 : |X(t)|=r\}$, and, as usual, set $T_r=\infty$ if $|X(t)|\neq r$ for all $t>0$. The *hitting measure* and *Green's function* of S_r are respectively the quantities,

$$H_r(x, d\xi) = P_x(X(T_r) \in d\xi, T_r < \infty)$$

and $g_r(x, y)$, where $g_r(x, y)$ is the density of the measure

$$\int_0^\infty P_x(T_r > t, X(t) \in dy) \, dt.$$

The hitting probability of S_r is $\Phi_r(x)=P_x(T_r<\infty)$. Our purpose in this paper is to explicitly compute these as well as some related quantities.

In brief we will do the following. In §2 we introduce the radial process $Z_\alpha(t)$ and use it to compute $\Phi_r(x)$ by the relation $\Phi_r(x)=P_{|x|}(\tau_r<\infty)$, where $\tau_r= \inf\{t>0 : Z_\alpha(t)=r\}$. The problem is trivial if $\alpha \leqq 1$ (see Proposition 2.1) since $\{r\}$ is a polar set for $Z_\alpha(t)$ in that case. Also, if the process is recurrent, then $\Phi_r(x)\equiv 1$ so the only cases of interest are $1<\alpha<N$. Our technique here is simply to note that

$$\Phi_r(x) = \frac{u(x, r)}{u(r, r)},$$

where $u(x, r)$ is the potential kernel of $Z_\alpha(t)$, and to compute $u(x, y)$. But having $u(x, y)$ one may explicitly compute more elaborate probabilities, e.g.,

$$P_x\Big(\operatorname*{Min}_{1\leqq i\leqq n} T_{r_i} = T_{r_j}\Big).$$

Some of these computations will be also carried out in §2. In §3 we compute $H_r(x, d\xi)$ by the method devised by M. Riesz [8] of inversion in an appropriate sphere, and in §4 we use the results of §2 and 3 to write down the Green's function of S_r.

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Previously, the above quantities were computed for the solid ball by Blumenthal, Getoor, and Ray in [3] by the use of Riesz's inversion technique, and in [7] the author computed these quantities for arbitrary finite sets in the case of recurrent one-dimensional stable processes with exponent $\alpha > 1$.

2. **The radial process.** Since $X(t)$ is isotropic, the process $Z_\alpha(t) = |X(t)|$ is a Markov process, and since $X(t)$ is a Feller process, it must be that $Z_\alpha(t)$ is also a Feller process. Thus by §9 of [1], there is a realization of $Z_\alpha(t)$ as a standard Markov process. We henceforth assume that $Z_\alpha(t)$ is this version of the process. If $\alpha = 2$, then $X(t)$ is Brownian motion, and it is well known (see [6], p. 60 or [2], §4) that the transition function of $Z_2(t)$ is given by

$$P_x(Z_2(t) \in A) = \int_A f_2(t, x, y)\mu(dy)$$

where $\mu(dy) = 2^{-N/2}[\Gamma((N/2)+1)]^{-1}y^N\,dy$, and

$$f_2(t, x, y) = \Gamma\left(\frac{N}{2}\right)(2t)^{-1}\left(\frac{xy}{2}\right)^{1-N/2} \exp\left[-(x^2+y^2)/4t\right]I_{N/2-1}\left(\frac{xy}{2t}\right), \tag{2.1}$$

where I_ν is the usual modified Bessel function.

Let T_β be the stable subordinator of exponent β, $0 < \beta < 1$, with $T_\beta(0) = 0$. Then it is a familiar fact that Z_α and $Z_2(T_{\alpha/2}(t))$ are equivalent provided that $T_{\alpha/2}$ and Z_2 are independent. Let $h_{\alpha/2}(t, u)$ denote the density function of $T_{\alpha/2}$. Then the transition function of Z_α is given by

$$P_x(Z_\alpha(t) \in A) = \int_A f_\alpha(t, x, y)\mu(dy)$$

where μ was defined above and

$$f_\alpha(t, x, y) = \int_0^\infty h_{\alpha/2}(t, u) f_2(u, x, y)\,du. \tag{2.2}$$

Let $\tau_r = \inf\{t > 0 : Z_\alpha(t) = r\}$ ($= \infty$ if $Z_\alpha(t) \neq r$ for all $t > 0$). It is clear that $\{r\}$ is a polar set for the radial process $Z_\alpha(t)$ if and only if the sphere $S_r = \{y : |y| = r\}$ is a polar set for the process $X(t)$. In addition, if r is a regular point of $\{r\}$ for the radial process, then all points on the sphere S_r are regular for this sphere for the process $X(t)$. The following fact ensues from Corollary 4.3 and Theorem 3.1 of [2]. For completeness, we sketch below an alternate proof which avoids the use of Hunt's capacity theory.

PROPOSITION 2.1. *For the radial process $Z_\alpha(t)$, r is regular for $\{r\}$ provided $\alpha > 1$. If $\alpha \leqq 1$, then $\{r\}$ is polar.*

Proof. Let $A_n = \{x \in R^1 : |x - r| < 1/n\}$ and let τ_n be the first hitting time of A_n. Set

$$H^\lambda_{A_n}(x, B) = E_x(\exp(-\lambda\tau_n)1_B(x(\tau_n));\ \tau_n < \infty),$$

and let $u^\lambda(x, y)$ be the Laplace transform of $f_\alpha(t, x, y)$. Then the usual first passage arguments show that

$$u^\lambda(x, r) = \int_{\bar{A}_n} H^\lambda_{A_n}(x, dz) u^\lambda(z, r). \tag{2.3}$$

Simple computations (see the proof of Corollary 4.3 of [2] for details) show that if $r>0$, then

$$f_\alpha(t, r, r) \sim kt^{-1/\alpha}, \qquad t \to 0,$$

where k is some constant (dependent on r)>0. Thus $u^\lambda(x, r) \to \infty$ as $x \to r$ when $\alpha \leqq 1$, while for $\alpha > 1$, $u^\lambda(x, r)$ is bounded and continuous in x in a neighborhood of r.

Suppose $\alpha \leqq 1$. Then (2.3) shows that

$$\infty > u^\lambda(x, r) \geqq E_x(\exp(-\lambda\tau_r); \tau_r < \infty) \inf_{z \in A_n} u^\lambda(z, r),$$

and it follows that $P_x(\tau_r < \infty) = 0$ for all $x \neq r$. Since

$$P_r(\tau_r < \infty) = \lim_{t \downarrow 0} \int_{R^N} f_\alpha(t, r, y) P_y(\tau_r < \infty)\, dy$$

we see that $\{r\}$ is polar for all r.

Now suppose $\alpha > 1$. Since $Z_\alpha(t)$ is a standard process it is quasi-left continuous (see [1], §9 for a definition) and thus $\tau_n \uparrow \tau_r$ and $X(\tau_n) \to X(\tau_r)$, a.s. P_x, $x \neq r$. It then follows from (2.3) that for $x \neq r$ and $r > 0$,

$$u^\lambda(x, r)/u^\lambda(r, r) = E_x(\exp(-\lambda\tau_r), \tau_r < \infty). \tag{2.4}$$

Hence

$$\lim_{x \to r} E_x(\exp(-\lambda\tau_r), \tau_r < \infty) = 1, \tag{2.5}$$

and it follows easily from this that r is regular for $\{r\}$ whenever $r > 0$. This completes the proof.

In view of the above result we shall henceforth only consider the processes with $\alpha > 1$. If the processes are recurrent, then $\Phi_r(x) \equiv 1$, so we need only consider transient processes (i.e., $\alpha < N$).

THEOREM 2.1. *Assume* $1 < \alpha < N$. *Then for* $r > 0$,

(2.6)

$$\Phi_r(x) = \frac{\pi^{1/2}\Gamma\left(\frac{\alpha+N}{2} - 1\right) 2^{2-\alpha}}{\Gamma\left(\frac{\alpha-1}{2}\right)} r^{N-\alpha} \left| |x|^2 - r^2 \right|^{(\alpha/2)-1} P^{1-N/2}_{-\alpha/2}\left(\frac{|x|^2 + r^2}{\left||x|^2 - r^2\right|}\right) (|x| r)^{1-N/2},$$

where P^ν_μ *is the usual Legendre function of the first kind.*

Proof. Since the processes are transient, $u^\lambda(x, y) \to u(x, y)$, $\lambda \downarrow 0$, where $u(x, y)$ is the potential kernel of $Z_\alpha(t)$. From (2.4) we see that

$$P_x(\tau_r < \infty) = u(x, r)/u(r, r), \tag{2.7}$$

so to establish (2.6) we need to compute the right-hand side of (2.7). This will be done in the following two lemmas.

LEMMA 2.1. *If $1 < \alpha < N$, then*

$$u_\alpha(x, y) = \frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-\alpha}{2}\right)2^{(N/2)-\alpha}}{\Gamma(\alpha/2)}(xy)^{1-N/2}|x^2-y^2|^{(\alpha/2)-1}P^{1-N/2}_{-\alpha/2}\left(\frac{x^2+y^2}{|x^2-y^2|}\right), \tag{2.8}$$

where P^ν_μ is the usual Legendre function of the first kind.

Proof. The stable subordinator $T_{\alpha/2}$ of exponent $\alpha/2$ is the unique stable process on $(0, \infty)$ whose transition density has Laplace transform

$$\int_0^\infty h_{\alpha/2}(t, u)e^{-\gamma u}\, du = \exp(-t\gamma^{\alpha/2}),$$

and thus

$$\int_0^\infty \int_0^\infty h_{\alpha/2}(t, u)e^{-\gamma u}\, du\, dt = \gamma^{-\alpha/2}.$$

Hence the potential kernel of $T_{\alpha/2}$ is $\Gamma(\alpha/2)^{-1}u^{(\alpha/2)-1}$. From (2.2) we then see that

$$u_\alpha(x, y) = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty u^{(\alpha/2)-1}f_2(u, x, y)\, du.$$

Using the explicit formula for $f_2(u, x, y)$ and formula 8, p. 196 of [4], we obtain (2.8).

LEMMA 2.2. *If $1 < \alpha < N$, then for $r > 0$,*

$$u_\alpha(r, r) = \frac{\pi^{-1/2}2^{\alpha-2}\Gamma\left(\frac{\alpha-1}{2}\right)}{\Gamma\left(\frac{\alpha+N}{2}-1\right)}\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-\alpha}{2}\right)2^{(N/2)-\alpha}}{\Gamma(\alpha/2)}r^{\alpha-N}. \tag{2.9}$$

Proof. This follows from (2.8) and the asymptotic relation (see formula 20, p. 164 of [5]),

$$P^{1-N/2}_{-\alpha/2}\left(\frac{x^2+y^2}{|x^2-y^2|}\right) \sim \frac{\pi^{-1/2}2^{\alpha-2}\Gamma\left(\frac{\alpha-1}{2}\right)}{\Gamma\left(\frac{\alpha+N}{2}-1\right)}y^{\alpha-2}|x^2-y^2|^{1-\alpha/2}, \qquad x \to y.$$

COROLLARY 2.1. *Assume $1 < \alpha < N$. Then for any $r > 0$,*

$$\Phi_r(0) = \frac{\Gamma\left(\frac{\alpha+N}{2}-1\right)\pi^{1/2}2^{2-\alpha}}{\Gamma(N/2)\Gamma\left(\frac{\alpha-1}{2}\right)}. \tag{2.10}$$

Proof. This follows from (2.6) and the asymptotic formula (see formula 3 of p. 163 of [5]) that for $y>0$,

$$P^{1-N/2}_{-\alpha/2}\left(\frac{x^2+y^2}{|x^2-y^2|}\right) \sim \frac{y^{(1-N/2)}}{\Gamma(N/2)} x^{-(1-N/2)}, \qquad x \to 0.$$

Corollary 2.2. *Assume $1<\alpha<N$. Then the capacity of the sphere of radius r is C_r where*

$$C_r = \frac{4\Gamma\left(\frac{\alpha+N}{2}-1\right)\pi^{(N+1)/2}\Gamma(\alpha/2)}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{\alpha-1}{2}\right)\Gamma\left(\frac{N-\alpha}{2}\right)} r^{N-\alpha}. \tag{2.11}$$

Proof. Let $\pi_r(d\xi)$ be the capacitory measure of the sphere $S_r=\{x : |x|=r\}$. The potential kernel of $X(t)$ is just the Riesz kernel $K|x|^{\alpha-N}$, where

$$K = \frac{\Gamma((N-\alpha)/2)}{4^{\alpha/2}\pi^{N/2}\Gamma(\alpha/2)},$$

and it is a basic fact that (see Chapter 6 of [1]) the capacitory potential of S_r is just $\Phi_r(x)$, i.e.,

$$\Phi_r(x) = \int_{S_r} K|y-x|^{\alpha-N}\pi_r(dy).$$

Thus $\Phi_r(0)=r^{\alpha-N}K\pi(S_r)$. Since $C_r=\pi(S_r)$, (2.11) follows from (2.10) and the above relation.

For $\alpha=2$ (i.e., Brownian motion) and $N>2$ the formula for $u(x, y)$ is considerably simpler.

$$u(x, y) = 2^{(N/2)-2}\Gamma(N/2)(N/2-1)^{-1}[\mathrm{Max}\,(x, y)]^{2-N}. \tag{2.12}$$

Using this, it is easily seen that Theorem 2.1 yields the well-known result

$$\begin{aligned}\Phi_r(x) &= 1, & x &\leqq r,\\ &= (x/r)^{2-N}, & x &> r.\end{aligned}$$

By the same type of arguments we may compute more elaborate hitting probabilities for the processes with $1<\alpha<N$. Let $B=\{r_1, r_2, \ldots, r_n\}$ where $r_1<r_2<\cdots<r_n$. Since potential $\sum_{i=1}^n u(x, r_i)\mu_i$ on N uniquely determines the numbers μ_i, we see that the matrix $U_{ij}=u(r_i, r_j)$ is invertible. Denote its inverse by $K_B(i, j)$. If τ_B is the first hitting time of B by the radial process, and if

$$T_B = \inf\{t > 0 : |X(t)| \in B\}$$

then, of course, $P_x(T_B<\infty)=P_{|x|}(\tau_B<\infty)$. However, it is a fundamental fact in the theory of Markov processes (see [1, Chapter 6]) that there is a bounded measure π having support on B such that

$$P_a(\tau_B < \infty) = \sum_{j=1}^{n} u(a, r_j)\pi_j. \tag{2.13}$$

[This fact may also be proved directly for the $Z_\alpha(t)$ process by an argument very similar to that used to deduce (2.7).] Since every point of B is regular for B, we see that π is the unique measure on B such that $1=(U\pi)_j$, $1\leqq j\leqq n$, and thus $\pi_j=\sum_{i=1}^n K_B(i,j)$. Consequently

$$P_a(\tau_B < \infty) = \sum_{i=1}^n \sum_{j=1}^n u(a, r_j)K_B(i, j). \tag{2.14}$$

In a similar manner we see that

$$P_x\Big(\operatorname*{Min}_{1\leqq i\leqq n} (T_{r_i}) = T_{r_j}, T_B < \infty\Big) = P_{|x|}(Z_\alpha(\tau_B) = r_j, \tau_B < \infty).$$

Set

$$H_B(a, r_i) = P_a(Z_\alpha(\tau_B) = r_i, \tau_B < \infty).$$

Then the $H_B(a, r_i)$ are uniquely determined by the equations

$$u(a, r_j) = \sum_{i=1}^n H_B(a, r_i)u(r_i, r_j), \qquad 1 \leqq j \leqq n,$$

and thus

$$H_B(a, r_j) = \sum_{i=1}^n u(a, r_i)K_B(r_i, r_j). \tag{2.15}$$

For a two point set $B=\{r_1, r_2\}$

$$K_B = \frac{1}{\Delta}\begin{pmatrix} U_{22} & -U_{12} \\ -U_{12} & U_{11}\end{pmatrix}.$$

where $\Delta=U_{11}U_{22}-(U_{12})^2$. Equations (2.14) and (2.15) then yield

$$\begin{aligned} P_x(T_B < \infty) &= \frac{u(|x|, r_1)u(r_2, r_2)+u(|x|, r_2)u(r_1, r_1)}{u(r_1, r_1)u(r_2, r_2)-u(r_1, r_2)^2} \\ &\quad -\frac{u(r_1, r_2)[u(|x|, r_1)+u(|x|, r_2)]}{u(r_1, r_1)u(r_2, r_2)-u(r_1, r_2)^2}, \end{aligned} \tag{2.16}$$

and

$$P_x(T_{r_1} < T_{r_2}) = \frac{u(|x|, r_1)u(r_2, r_2)-u(|x|, r_2)u(r_2, r_1)}{u(r_1, r_1)u(r_2, r_2)-u(r_1, r_2)^2}, \tag{2.17a}$$

$$P_x(T_{r_2} < T_{r_1}) = \frac{u(|x|, r_2)u(r_1, r_1)-u(|x|, r_1)u(r_1, r_2)}{u(r_1, r_1)u(r_2, r_2)-u(r_1, r_2)^2}. \tag{2.17b}$$

In particular, for $\alpha=2$ we obtain the following well-known results for Brownian motion in dimension $N\geqq 3$.

$$\begin{aligned} P_x(T_{r_1} < T_{r_2}) &= 1, & |x| &\leqq r_1, \\ &= 0, & |x| &\geqq r_2, \\ &= \frac{|x|^{2-N}-r_2^{2-N}}{r_1^{2-N}-r_2^{2-N}}, & r_1 \leqq |x| &\leqq r_2, \end{aligned}$$

and

$$\begin{aligned} P_x(T_{r_2} < T_{r_1}) &= 0, & |x| &\leqq r_1, \\ &= |x/r_2|^{2-N}, & |x| &\geqq r_2, \\ &= \frac{r_1^{2-N}-|x|^{2-N}}{r_1^{2-N}-r_2^{2-N}}. & r_1 \leqq |x| &\leqq r_2. \end{aligned}$$

In the above discussion we omitted those processes with $\alpha \geqq N$. We will now fill in this detail. If $N=1$, then since $\alpha>1$, the processes are point recurrent, and the above methods are not directly applicable since $u(x, y)=\infty$. However, in this case a sphere consists of two points, and explicit formulas for the hitting distribution of finite sets were given in [7] (see §3). Alternately, it is easily seen that the recurrent potential kernel of the $Z_\alpha(t)$ process is given by $u(x, y)=a(y-x)+a(y+x)$ where $a(x)$ is the recurrent potential kernel of $X(t)$ given in [7]. With this u, the hitting distribution is again given by (2.15). The remaining process is $\alpha=N=2$, i.e., planar Brownian motion. Owing to the continuity of the paths, the hitting probabilities for a finite B can be reduced to that of a two point set. But for such a set the result is well known. (See, e.g., [6, p. 62].)

3. **The hitting measure of S_r.** Assume $1<\alpha<N$. It is intuitively clear that the capacitory measure $\pi_r(d\xi)$ of S_r is $C_r\, d\sigma_r(\xi)$, where here and in the following, σ_r is the uniform measure on S_r, and C_r is the capacity of S_r given in (2.11). To establish this fact rigorously we note that since every point of S_r is regular for S_r, the measure π_r is the unique bounded measure having support on S_r such that for all $x \in S_r$,

$$1 = K \int_{S_r} |\xi - x|^{\alpha-N} \pi_r(d\xi), \tag{3.1}$$

where here and in the following,

$$K = \frac{\Gamma((N-\alpha)/2)}{4^{\alpha/2}\pi^{N/2}\Gamma(\alpha/2)}. \tag{3.2}$$

A change to spherical coordinates now easily shows that $C_r\, d\sigma_r(\xi)$ satisfies (3.1).

The main result of this section is the following

THEOREM 3.1. *Assume $1<\alpha<N$. Then the hitting measure $H_r(x, d\xi)$ of S_r is given by*

(3.3)

$$H_r(x, d\xi) = \frac{\Gamma\left(\frac{\alpha+N}{2}-1\right)\pi^{1/2}2^{2-\alpha}r^{N-\alpha}}{\Gamma(N/2)\Gamma\left(\frac{\alpha-1}{2}\right)} \,\big|\, |x|^2 - r^2 \big|^{\alpha-1} |\xi - x|^{2-\alpha-N}\, d\sigma_r(\xi), \quad |x| \neq r,$$

while $H_r(x, d\xi)$ is the unit mass at x if $|x|=r$.

REMARK. The basis of the computation of $H_r(x, d\xi)$ which we shall use here is that of inversion in an appropriate sphere orthogonal to S_r. This idea was first used by M. Riesz in [8] to compute (in probabilistic terminology) the hitting measure of the solid ball. Later, Blumenthal, Getoor, and Ray [3] extended these computations to complete the story for the ball.

Proof. Consider first the case when $|x|>r$. The inversion in the sphere $\{y : |y-x|=a\}$ is the change of variable $y \to y' = x + a^2(y-x)|y-x|^{-2}$. Choose $a^2 = |x|^2 - r^2$. Then the sphere S_r and the inverting sphere are orthogonal, and thus the transformation maps S_r onto S_r. If y', z' are the images of y, z under this inversion, then

$$|y'-z'| = \frac{a^2|y-z|}{|y-x|\,|z-x|}. \tag{3.4}$$

Define a measure $\mu_x(d\xi)$ on S_r by

$$\mu_x(d\xi)|\xi-x|^{\alpha-N} = \pi_r(d\xi') \tag{3.5}$$

where ξ' is the image of ξ, and π_r is the capacitory measure of S_r. If $z \in S_r$ then so does z', and (3.1), (3.4), and (3.5) now show that if $z \in S_r$

$$1 = K\int_{S_r} |z'-y'|^{\alpha-N}\pi_r(dy') = K\left[\frac{a^2}{|z-x|}\right]^{\alpha-N}\int_{S_r} |z-y|^{\alpha-N}\mu_x(dy).$$

Thus if $z \in S_r$,

$$K|z-x|^{\alpha-N} = K(a^2)^{\alpha-N}\int_{S_r} K|z-y|^{\alpha-N}\mu_x(dy). \tag{3.6}$$

But since every point of S_r is regular for S_r, the measure $H_r(x, d\xi)$ is the unique measure supported on S_r such that

$$K|z-x|^{\alpha-N} = \int_{S_r} H_r(x, d\xi)K|z-\xi|^{\alpha-N}, \qquad z \in S_r. \tag{3.7}$$

Thus

$$H_r(x, d\xi) = K(a^2)^{\alpha-N}\mu_x(d\xi). \tag{3.8}$$

Suppose $\mu_x(d\xi) = k_x(\xi)\, d\sigma_r(\xi)$. Then (3.5) shows that

$$k_x(\xi) = C_r|\xi-x|^{N-\alpha}\, d\sigma(\xi')/d\sigma(\xi).$$

However, it is clear from the geometry that

$$\frac{d\sigma(\xi')}{|\xi'-x|^{N-1}} = \frac{d\sigma(\xi)}{|\xi-x|^{N-1}},$$

and thus, using (3.4), we find that

$$k_x(\xi) = C_r(a^2)^{N-1}|\xi-x|^{2-\alpha-N},$$

and thus by (3.8),

$$H_r(x, d\xi) = KC_r\,\big||x|^2-r^2\big|^{\alpha-1}|\xi-x|^{2-\alpha-N}\, d\sigma_r(\xi), \qquad |x| > r. \tag{3.9}$$

Suppose now that $|x|<r$. An inversion in the sphere S_r sends x to $x' = r^2x/|x|^2$,

and by what has just been shown above we know that $H_r(x', d\xi)$ satisfies the equation

$$K|y-x'|^{\alpha-N} = \int_{S_r} K^2 C_r\,||x'|^2-r^2|^{\alpha-1}|\xi-x'|^{2-\alpha-N}|\xi-y|^{\alpha-N}\,d\sigma_r(\xi), \quad y \in S_r.$$

But

$$|y-x'| = r|y-x|/|x|, \qquad [|x'|^2-r^2] = r^2[r^2-|x|^2]/|x|^2,$$

and thus we find that

$$H_r(x, d\xi) = KC_r\,||x|^2-r^2|^{\alpha-1}|\xi-x|^{2-\alpha-N}\,d\sigma_r(\xi)$$

satisfies (3.7). This completes the proof.

We note that for $\alpha=2$, i.e., Brownian motion, the kernel in (3.3) becomes the classical Poisson kernel (as it should), and that for a general α, the kernel is a very close analogue of this classical kernel.

We conclude this section with a comment on the quantity $H_r(x, d\xi)$ in the case of a recurrent stable process, $\alpha \geqq N$. For $\alpha=N=2$, i.e., planar Brownian motion, it is well known that (3.3) still gives the correct result. For $\alpha>1=N$, the sphere is a two point set, and an explicit formula for $H_r(x, d\xi)$ was computed in [7, §3].

4. **The Green's function.** Again we consider the case when $1<\alpha<N$. The Green's function $g_r(x, y)$ for the sphere $S_r=\{y : |y|=r\}$ is uniquely defined by

$$g_r(x, y) \equiv K|y-x|^{\alpha-N}-K^2C_r\int_{S_r}||x|^2-r^2|^{\alpha-1}|\xi-x|^{2-\alpha-N}|\xi-y|^{\alpha-N}\,d\sigma_r(\xi) \tag{4.1}$$

where K is given in (3.2) and C_r is the capacity given in (2.11). Set

$$I = \int_{S_r}|\xi-x|^{2-\alpha-N}|\xi-y|^{\alpha-N}\,d\sigma_r(\xi).$$

Consider the case when $|x|>r$. An inversion in the sphere $\{y : |y-x|^2=|x|^2-r^2\}$ sends $\xi \to \xi' \in S_r$ and $y \to y'$. Performing this change of variable we find that

$$\begin{aligned} I &= (|x|^2-r^2)^{1-N}|y'-x|^{N-\alpha}\int_{S_r}|\xi'-y'|^{\alpha-N}\,d\sigma_r(\xi') \\ &= (|x|^2-r^2)^{1-N}|y'-x|^{N-\alpha}\Phi_r(y')(KC_r)^{-1} \\ &= (KC_r)^{-1}(|x|^2-r^2)^{1-\alpha}|y-x|^{\alpha-N}\Phi_r(y'). \end{aligned}$$

Substituting this expression for I into (4.1) shows that

$$g_r(x, y) = K|y-x|^{\alpha-N}[1-\Phi_r(y')], \qquad |x| > r. \tag{4.2}$$

Now a simple computation shows that

$$|y'|^2|y-x|^2 = |x|^2|y|^2+r^4-2r^2(x\cdot y) = |y|^2|x-r^2y/|y|^2|^2, \tag{4.3}$$

and thus for $|x|>r$

$$g_r(x, y) = K|y-x|^{\alpha-N}\left\{1-\Phi_r\left(\frac{y}{|y-x|}\,|x-r^2y/|y|^2|\right)\right\}.$$

To compute $g_r(x, y)$ for $|x|<r$, note that an inversion in the sphere S_r sends $x \to \bar{x}=r^2x/|x|^2$, $y \to \bar{y}=r^2y/|y|^2$ and that $|\bar{x}|>r$. Using (4.1) and some simple computations we easily obtain that

$$g_r(x, y)(r^2/|x|\,|y|)^{\alpha-N} = g_r(\bar{x}, \bar{y}). \tag{4.4}$$

Thus for $|x|<r$,

$$\begin{aligned} g_r(x, y) &= K|y-x|^{\alpha-N}\{1-\Phi_r((\bar{y})')\} \\ &= K|y-x|^{\alpha-N}\left\{1-\Phi_r\left(\frac{\bar{y}}{|\bar{y}-\bar{x}|}\,|\bar{x}-\bar{y}r^2/|\bar{y}|^2|\right)\right\} \\ &= K|y-x|^{\alpha-N}\left\{1-\Phi_r\left(\frac{y|x|}{|y-x|\,|y|}\,|y-xr^2/|x|^2|\right)\right\} \\ &= K|y-x|^{\alpha-N}\left\{1-\Phi_r\left(\frac{y}{|y-x|}\,|x-yr^2/|y|^2|\right)\right\}, \end{aligned}$$

where the last equality follows from the symmetry of $g_r(x, y)$ and the fact that $\Phi_r(t)$ is a function of $|t|$. Combining the above results we obtain

THEOREM 4.1. *The Green's function of the sphere is given by*

$$g_r(x, y) = K|y-x|^{\alpha-N}\left\{1-\Phi_r\left(\frac{y}{|y-x|}\,|x-yr^2/|y|^2|\right)\right\}, \tag{4.5}$$

where Φ_r is the hitting probability given in (2.6).

For $\alpha=2$, $N>2$, the above Green's function is the classical one for the Laplacian. To see this, note that the first and second equality in 4.3 and a little computation shows that for $|x|>r$,

$$[|y'|^2-r^2]|y-x|^2 = [|x|^2-r^2][|y|^2-r^2].$$

Hence for $|x|>r$, $|y'|>r$ iff $|y|>r$. It follows that $g_r(x, y)=0$ if either $|x|>r$, $|y|\leqq r$ or $|x|\leqq r$, $|y|>r$, while for $|x|<r$, $|y|<r$ or $|x|>r$, $|y|>r$

$$g_r(x, y) = K|y-x|^{2-N}-K|y/r|^{2-N}|x-r^2y/|y|^2|^{2-N}.$$

For $\alpha=N=2$, i.e.,.planar Brownian motion, the Green's function of the circle is just the classical Green's function for the Laplacian, and may be found in all books on partial differential equations. For $\alpha>1=N$, the Green's function of the sphere was computed in [7].

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